

THE DECOMPOSABILITY OF TORSION FREE MODULES OF FINITE RANK

BY
EBEN MATLIS

Throughout this paper R will be an integral domain, not a field; Q will be the quotient field of R ; and K will be the R -module Q/R . R will be said to have *property D*, if every torsion-free R -module of finite rank is a direct sum of R -modules of rank one.

In [9, §6] I attempted to give necessary and sufficient conditions for an integral domain to have property D . I have since discovered, however, that Lemma 6.4 of that paper is false, with the result that Theorem 6.1 is also false as stated, and needs some rather strong additional hypotheses to be resurrected. Other false lemmas based on Lemma 6.4 are Proposition 6.5 and Corollary 6.7; and Lemma 6.8 remains without a valid proof.

Compounding the confusion, I have also discovered an error in Kaplansky's proof of his theorem [4] which purported to handle the Noetherian case, and this theorem remains without a valid proof. Proposition 6.11 and Corollary 6.12 of my paper [9] were based on Kaplansky's proof, and are false as stated. Corollary 6.13 remains without a valid proof.

The purpose of this paper is to attempt to restore some order out of this chaos. In Theorem 4 I have settled the Noetherian case. This theorem states that a Noetherian domain has property D if and only if it is one of two types of Noetherian domains. However, I have not been able to find an example of a ring of type (1); i.e. a principal ideal domain R with exactly two nonzero prime ideals P_1, P_2 and such that R_{P_1}, R_{P_2} are complete discrete valuation rings. It was precisely this type of ring that Kaplansky thought that he had proved could not exist. Thus one of the remaining unsolved tasks in finding all of the kinds of integral domains that have property D is either to produce an example of this type of ring, or to prove that it cannot exist.

In Theorem 3 I have shown that if R is an h -local domain with more than one maximal ideal, then R has property D if and only if R has exactly two maximal ideals M_1, M_2 and R_{M_1}, R_{M_2} are maximal valuation rings. The question of whether such a ring can exist is, of course, a generalization of the previous question raised.

Theorem 2 proves that a valuation ring has property D if and only if it is a maximal valuation ring. The major part of this result, namely that a maximal valuation ring has property D , is originally due to Kaplansky [3, Theorem 12] who generalized Prüfer's result [11] for complete discrete valuation rings. Both of

Received by the editors April 4, 1967.

these authors proved the far stronger result that for the kinds of rings they were considering a torsion-free module of countable rank is a direct sum of modules of rank one.

Theorem 1 is a partial result which states that if a local domain has property D , then it must be complete in the R -topology. However, only in the Noetherian case have I been able to find necessary and sufficient conditions for a local domain to have property D . As indicated above, modulo the question of existence, I have solved the problem for h -local domains with more than one maximal ideal. Thus in order to solve the problem for all h -local domains only the case of a single maximal ideal, i.e. a local domain, remains to be settled.

The rings of type (2) in Theorem 4 provide examples of local domains with property D . This type of ring is a complete Noetherian local domain of Krull dimension one such that every ideal can be generated by two elements. The proof that a Noetherian local domain has property D if and only if it is of this type is based on an earlier result of the author [7, Corollary 3] and on a theorem of Bass [1, Theorem 1.7]. The existence of rings of type (2) proves that Theorem 6.1 of [9] is false as stated. An indication of the difficulty there will be in finding necessary and sufficient conditions for a local domain to have property D is given by the fact that these conditions must be sufficiently general to include both maximal valuation rings and rings of type (2).

Theorem 5 attempts to handle the integrally closed case. In the presence of a strong additional hypothesis Theorem 5 gives necessary and sufficient conditions for an integrally closed domain to have property D . With these additional assumptions Theorem 6.1 of [9] has been resurrected. If it could be shown that a Prüfer ring R with two maximal ideals and property D must have $\text{inj. dim.}_R R = 1$, then Theorem 5 would hold for an arbitrary integrally closed domain, and this case would be solved. This conjecture on Prüfer rings seems reasonable since it holds in the h -local case, but I am beginning to doubt if it is true.

For the sake of convenience, we will accumulate all of the necessary definitions in the following paragraphs.

DEFINITIONS. (1) A *local ring* is a ring with a single maximal ideal; the ring need not be Noetherian. Some authors refer to such a ring as a quasi-local ring.

(2) An *h -local ring* is a ring such that each nonzero ideal is contained in only a finite number of maximal ideals, and such that each nonzero prime ideal is contained in only a single maximal ideal. A local ring is, of course, an h -local ring; but so is every Noetherian integral domain of Krull dimension one.

(3) A *Prüfer ring* is an integral domain such that every finitely generated ideal is projective. It is easily seen that an integral domain R is a Prüfer ring if and only if R_M is a valuation ring for every maximal ideal M of R .

(4) If A is an R -module, we will use the notation $\text{inj. dim.}_R A = 1$ to signify that A has injective dimension one as a module over R . Thus, in particular, $\text{inj. dim.}_R R = 1$ if and only if R is an injective R -module.

(5) The *Krull dimension* of a Noetherian domain is the maximal length of a chain of prime ideals in the domain.

(6) A module over an integral domain is called *h-divisible* if it is a homomorphic image of an injective module.

(7) A module A over an integral domain R is called a *cotorsion module* if $\text{Hom}_R(Q, A) = 0$ and $\text{Ext}_R^1(Q, A) = 0$.

(8) Every integral domain R has a natural topology, called the *R-topology*, obtained by taking the nonzero ideals as the neighborhoods of zero. This topology is a uniform topology and so R has a completion; this completion is a commutative ring. We have by [8, Theorem 5.4 and Proposition 6.4] that the following statements are equivalent:

(i) R is complete in the R -topology.

(ii) $R \cong \text{Hom}_R(K, K)$.

(iii) R is a cotorsion R -module.

(9) A valuation ring is called a *maximal valuation ring* if R is complete in the R -topology and if $\text{inj. dim.}_R R = 1$ (see [6]).

The following two lemmas were proved in [9], but we include their statements here for the sake of the coherence of the presentation.

LEMMA 1. *Let R be an integral domain with property D , and let S be a subring of Q containing R . Then S also has property D .*

Proof. See [9, Lemma 6.2].

LEMMA 2. *Let R be an integral domain, complete in the R -topology. Then $K \otimes_R I$ has no proper, nonzero, h -divisible submodules for each fractionary ideal I of R .*

Proof. See [9, Lemma 6.3].

THEOREM 1. *Let R be a local domain with property D . Then R is complete in the R -topology.*

Proof. Consider an exact sequence of the form

$$(*) \quad 0 \longrightarrow R \longrightarrow A \xrightarrow{p} Q \longrightarrow 0$$

where A is an R -module. Since A is a torsion-free R -module of rank two, we have by property D that $A = A_1 \oplus A_2$, where A_1, A_2 are torsion-free R -modules of rank one. Tensoring exact sequence $(*)$ with K we obtain the isomorphism:

$$K \cong (K \otimes_R A_1) \oplus (K \otimes_R A_2).$$

But K is an indecomposable R -module by [9, Corollary 4.2]. Therefore, we can assume that $K \otimes_R A_2 = 0$. This shows that $A_2 \cong Q$. Let f be the restriction of p to A_2 . Now f is nonzero; for if $f = 0$, then $A_2 \subset \text{Ker } p = R$, which is impossible, since no submodule of R is isomorphic to Q . Thus $f(A_2)$ is a nonzero divisible submodule

of Q , and hence $f(A_2) = Q$. Therefore, we have $A = R + A_2$. Since a nonzero homomorphism of one torsion-free R -module of rank one into another of the same kind is a monomorphism, we have $A_2 \cap \text{Ker } p = 0$; i.e. $A_2 \cap R = 0$. Hence $A = R \oplus A_2$, and exact sequence (*) is split. Since exact sequence (*) was an arbitrary representative of an element of $\text{Ext}_R^1(Q, R)$, we have that $\text{Ext}_R^1(Q, R) = 0$. Thus by [8, Corollary 6.11] R is complete in the R -topology.

LEMMA 3. *Let R be a domain, complete in the R -topology. Then R is a maximal valuation ring if and only if $\text{inj. dim.}_R I = 1$ for every nonzero ideal I of R .*

Proof. If R is a maximal valuation ring, then $\text{inj. dim.}_R I = 1$ for every nonzero ideal I of R by [6, Theorem 4].

Conversely, assume that this condition holds. Let a and b be two nonzero elements of R , and let $I = Ra \cap Rb$. Since $\text{inj. dim.}_R I = 1$, we have that Q/I is an injective R -module. By Lemma 2, Q/I is an indecomposable, injective R -module.

Let $x = 1 + I$ in Q/I . Since the annihilator of x is I , we have by [5, Theorem 2.4] that I is an irreducible ideal; i.e. I is not the intersection of two properly larger ideals. Thus either $I = Ra$ or $I = Rb$; and so either $Ra \subset Rb$ or $Rb \subset Ra$. Therefore, we have shown that R is a valuation ring. Since $\text{inj. dim.}_R R = 1$ and R is complete in the R -topology, R is a maximal valuation ring.

THEOREM 2. *Let R be a valuation ring. Then R is a maximal valuation ring if and only if R has property D .*

Proof. It is Kaplansky's contribution that a maximal valuation ring has property D [3, Theorem 12]. Conversely, assume that R has property D . Then R is complete in the R -topology by Theorem 1.

Let I and J be nonzero ideals of R , and consider an exact sequence of the form

$$(*) \quad 0 \longrightarrow I \longrightarrow A \xrightarrow{p} J \longrightarrow 0$$

where A is an R -module. Since A is a torsion-free R -module of rank two, we have by property D that $A = A_1 \oplus A_2$, where A_1, A_2 are torsion-free R -modules of rank one. Since R is a valuation ring, the R -submodules of J are linearly ordered. Thus we can assume that $p(A_1) \subset p(A_2)$. But $J = p(A_1) + p(A_2)$, and so $J = p(A_2)$. It follows that $A = I + A_2$. Since an R -homomorphism of one torsion-free R -module of rank one into another of the same kind is either zero or a monomorphism, we have $A_2 \cap \text{Ker } p = 0$; i.e. $A_2 \cap I = 0$. Thus $A = I \oplus A_2$, and so exact sequence (*) is split. Since exact sequence (*) was an arbitrary representative of an element of $\text{Ext}_R^1(J, I)$, we have that $\text{Ext}_R^1(J, I) = 0$. But $\text{Ext}_R^2(R/J, I) \cong \text{Ext}_R^1(J, I)$, and so $\text{Ext}_R^2(R/J, I) = 0$. Since J was an arbitrary ideal of R , this shows that $\text{inj. dim.}_R I = 1$ for every nonzero ideal I of R . Therefore R is a maximal valuation ring by Lemma 3.

The following lemma was proved in [9].

LEMMA 4. *Let R be an h -local domain with more than two maximal ideals. Then R has an indecomposable, torsion-free R -module of rank two.*

Proof. See [9, Proposition 6.9].

LEMMA 5. *Let R be an h -local domain with more than one maximal ideal. If R has property D , then $\text{inj. dim.}_R I = 1$ for every nonzero ideal I of R .*

Proof. Assume that R has property D , and let I be a nonzero ideal of R such that $\text{inj. dim.}_R I$ is not equal to one. Since R is h -local, we have by [9, Theorem 3.3] that

$$\text{inj. dim.}_R I = \sup. \text{inj. dim.}_{R_M} I_M$$

where M ranges over all of the maximal ideals of R . Thus there exists a maximal ideal M of R such that $\text{inj. dim.}_{R_M} I_M > 1$. It follows immediately from [2, Proposition 4.1.3] that we have $\text{inj. dim.}_R I_M > 1$ also. Therefore, there exists a nonzero ideal J of R such that $\text{Ext}_R^2(R/J, I_M) \neq 0$. Since $\text{Ext}_R^2(R/J, I_M) \cong \text{Ext}_R^1(J, I_M)$, we have $\text{Ext}_R^1(J, I_M) \neq 0$.

Thus we have an exact sequence:

$$(1) \quad 0 \longrightarrow I_M \longrightarrow A \xrightarrow{f} J \longrightarrow 0$$

(where A is a torsion-free R -module of rank two) which is not a split exact sequence. Since R has property D , $A = A_1 \oplus A_2$, where A_1, A_2 are torsion-free R -modules of rank one. We can assume that the projection g of I_M into A_1 is not zero. Since an R -homomorphism of one torsion-free R -module of rank one into another of the same kind is either zero or a monomorphism, we have that g is a monomorphism.

Thus we have an exact sequence:

$$(2) \quad 0 \longrightarrow I_M \xrightarrow{g} A_1 \longrightarrow B \longrightarrow 0$$

where B is a torsion R -module. From this exact sequence we derive the exact sequence:

$$(3) \quad \text{Hom}_R(B, J) \rightarrow \text{Hom}_R(A_1, J) \rightarrow \text{Hom}_R(I_M, J).$$

Since B is a torsion R -module, we have that $\text{Hom}_R(B, J) = 0$. Now R_M has property D by Lemma 1, and so R_M is complete in the R_M -topology by Theorem 1. Therefore R_M is a cotorsion R_M -module by [8, Corollary 6.11]. It follows from [8, Lemma 1.1] that I_M is also a cotorsion R_M -module, and thus $\text{Ext}_{R_M}^1(Q, I_M) = 0$. Since $\text{Ext}_R^1(Q, I_M) \cong \text{Ext}_{R_M}^1(Q, I_M)$ by [2, Proposition 4.1.3], we have that I_M is also a cotorsion R -module. By [8, Theorem 8.5] a complete h -local domain has to be a local domain. Therefore, since R is an h -local domain with more than one maximal ideal, R is not complete in the R -topology. Thus by [8, Corollary 6.11] R is not a cotorsion R -module. But then by [8, Proposition 5.11] an R -homomorphism from a cotorsion R -module into R must be zero. Therefore, we have $\text{Hom}_R(I_M, J) = 0$. Since the end modules of exact sequence (3) are zero, we must have $\text{Hom}_R(A_1, J) = 0$ also.

Returning to exact sequence (1) we see that $\text{Hom}_R(A_1, J) = 0$ implies that $f(A_1) = 0$. Thus $A_1 \subset \text{Ker } f = I_M$. It follows that $f(A_2) = J$, and so $I_M + A_2 = A$. Since A_2 and J are torsion-free R -modules of rank one, an R -homomorphism of A_2 into J is either zero or a monomorphism. Therefore, $A_2 \cap \text{Ker } f = 0$; i.e. $A_2 \cap I_M = 0$. Hence we have $A = I_M \oplus A_2$. This shows that exact sequence (1) is split, in contradiction to our choice of exact sequence (1) as a nonsplit sequence. This contradiction proves that $\text{inj. dim.}_R I = 1$ for every nonzero ideal I of R .

THEOREM 3. *Let R be an h -local domain with more than one maximal ideal. Then R has property D if and only if R has only two maximal ideals M_1, M_2 and R_{M_1}, R_{M_2} are maximal valuation rings. In this case R is a Prüfer ring.*

Proof. It was established in [9, Lemma 6.10] that if the conditions are satisfied, then R has property D . Hence let us assume that R has property D . By Lemma 4, R has only two maximal ideals M_1 and M_2 .

Let J be a nonzero ideal of R_{M_1} , and let $I = J \cap R$. By [9, Theorem 3.3] we have that

$$\text{inj. dim.}_R I = \sup \text{inj. dim.}_{R_{M_i}} I_{M_i}$$

for $i = 1, 2$. By Lemma 5, $\text{inj. dim.}_R I = 1$, and thus $\text{inj. dim.}_{R_{M_1}} I_{M_1} = 1$ also. Since $J = I_{M_1}$, we have $\text{inj. dim.}_{R_{M_1}} J = 1$. Thus every nonzero ideal of R_{M_1} has injective dimension one over R_{M_1} .

By Lemma 1, R_{M_1} has property D , and thus by Theorem 1, R_{M_1} is complete in the R_{M_1} -topology. Therefore, by Lemma 3, R_{M_1} is a maximal valuation ring. Similarly, R_{M_2} is a maximal valuation ring.

It is well known that if R_M is a valuation ring for every maximal ideal M of R , then R is a Prüfer ring.

THEOREM 4. *Let R be a Noetherian integral domain. Then R has property D if and only if R is one of the following two types of rings:*

(1) *R is a principal ideal domain with only two distinct, nonzero, prime ideals P_1, P_2 and R_{P_1}, R_{P_2} are complete discrete valuation rings.*

(2) *R is a complete Noetherian local domain of Krull dimension one, and every ideal of R can be generated by two elements.*

Proof. Assume that R is a ring of type (1). Since a complete discrete valuation ring is a maximal valuation ring, R has property D by Theorem 3. Assume that R is a ring of type (2). Then the integral closure of R is a finitely generated R -module (see [7, Theorem 4]). Hence by Bass' theorem [1, Theorem 1.7], every finitely generated torsion-free R -module is a direct sum of R -modules of rank one. But by [7, Corollary 3] every torsion-free R -module of finite rank is the direct sum of a divisible R -module and a finitely generated R -module. Thus R has property D .

Conversely, assume that R has property D . Suppose that R has three distinct prime ideals P_1, P_2, P_3 of rank one. Let $S = R - (P_1 \cup P_2 \cup P_3)$; then R_S has exactly three maximal ideals and is an h -local domain. By Lemma 1, R_S has property D .

But this contradicts Lemma 4. Hence R has at most two distinct prime ideals of rank one. In this case it is well known that R must have Krull dimension one, and that R has at most two maximal ideals.

A Noetherian domain of Krull dimension one is an h -local ring. Thus we have two cases to consider.

Case I. R has exactly two distinct maximal ideals M_1, M_2 .

By Theorem 3, R is a Prüfer ring. A Noetherian Prüfer ring with only two maximal ideals is a principal ideal domain. By Theorem 3, R_{M_1} and R_{M_2} are maximal valuation rings. But a Noetherian maximal valuation ring is a complete discrete valuation ring. Thus R is a ring of type (1).

Case II. R is a Noetherian local domain.

By Theorem 1, R is complete in the R -topology. Let M be the maximal ideal of R . Since R has Krull dimension one, the R -topology and the M -adic topology coincide. Therefore the integral closure of R is a finitely generated R -module (see [7, Theorem 4]). Thus we can apply Bass' theorem [1, Theorem 1.7] to conclude that every ideal of R can be generated by two elements, and so R is a ring of type (2).

LEMMA 6. *Let R be a Prüfer ring with at most two maximal ideals; and suppose that R_M is a maximal valuation ring for each maximal ideal M of R . Then the following statements are equivalent:*

- (1) R is an h -local ring.
- (2) $\text{inj. dim.}_R R = 1$.

Proof. Assume that R is an h -local ring, and let M_1 and M_2 be the maximal ideals of R (it does not matter in this case if $M_1 = M_2$). Since R_{M_i} is a maximal valuation ring, we have $\text{inj. dim.}_{R_{M_i}} R_{M_i} = 1$ for $i = 1, 2$. By [9, Theorem 3.3] we have that

$$\text{inj. dim.}_R R = \sup. \text{inj. dim.}_{R_{M_i}} R_{M_i}.$$

Thus $\text{inj. dim.}_R R = 1$.

Conversely, assume that $\text{inj. dim.}_R R = 1$. Suppose that R is not an h -local ring. Then, in particular, R has exactly two distinct maximal ideals M_1 and M_2 . Let $S = R_{M_1} + R_{M_2}$ in \mathcal{Q} . Since $R = R_{M_1} \cap R_{M_2}$, we have an exact sequence of R -modules:

$$0 \rightarrow R \rightarrow R_{M_1} \oplus R_{M_2} \rightarrow S \rightarrow 0.$$

Because R_{M_1} is complete in the R_{M_1} -topology, R_{M_1} is a cotorsion R_{M_1} -module by [8, Corollary 6.11]. It follows from [2, Proposition 4.1.3] that R_{M_1} is also a cotorsion R -module. Similarly R_{M_2} is a cotorsion R -module; and hence so is $R_{M_1} \oplus R_{M_2}$.

Let H be the completion of R in the R -topology. Since S is a torsion-free R -module, we have by [8, Proposition 2.8] that the above monomorphism $R \rightarrow R_{M_1} \oplus R_{M_2}$ can be extended to a monomorphism $H \rightarrow R_{M_1} \oplus R_{M_2}$. Thus we have an induced monomorphism $H/R \rightarrow S$. Now H/R is torsion-free and divisible by

[8, Proposition 5.2] and S is a torsion-free R -module of rank one, and so this monomorphism is either zero or an isomorphism. If it is an isomorphism, then $H \cong R_{M_1} \oplus R_{M_2}$, and R is an h -local ring by [8, Theorem 8.5]. This contradicts our assumption on R , and thus $H/R=0$; i.e. R is complete in the R -topology.

Because R is not an h -local ring there exists a nonzero prime ideal P of R such that $P \subset M_1 \cap M_2$. By Zorn's Lemma we can assume that P is maximal among prime ideals contained in $M_1 \cap M_2$. Let $V = R_P$; then $R_{M_1} \subset V$, and so V is a valuation ring. Since PV is the maximal ideal of V , we have that $PV \subset R_{M_1}$. Similarly $PV \subset R_{M_2}$, and thus $PV \subset R_{M_1} \cap R_{M_2} = R$. But $P = PV \cap R$, and so $P = PV$.

Let \bar{R} be the integral domain R/P , \bar{Q} the quotient field of \bar{R} , and $\bar{K} = \bar{Q}/\bar{R}$. Then $\bar{Q} = V/PV = V/P$, and so $\bar{K} \cong V/R$. We will identify \bar{K} with V/R whenever necessary. In particular, we have an exact sequence of R -modules:

$$(i) \quad 0 \rightarrow \bar{K} \rightarrow K \rightarrow Q/V \rightarrow 0.$$

Since $\text{inj. dim.}_R R = 1$, K is an injective R -module. Therefore, from the exact sequence (i) we derive an exact sequence:

$$(ii) \quad \text{Hom}_R(K, K) \rightarrow \text{Hom}_R(\bar{K}, K) \rightarrow 0.$$

We have an isomorphism $\text{Hom}_R(\bar{K}, K) \cong \text{Hom}_{\bar{R}}(\bar{K}, \bar{K})$. For let $f \in \text{Hom}_R(\bar{K}, K)$ and let $x \in \bar{K} = V/R$. Then $x = r/s + R$, where $r \in R$ and $s \in R - P$. Thus $sf(x) = 0$, which shows that $f(x) \in V/R = \bar{K}$. Hence we have $\text{Hom}_R(\bar{K}, K) \cong \text{Hom}_{\bar{R}}(\bar{K}, \bar{K})$. Since $P\bar{K} = 0$, it follows that $\text{Hom}_R(\bar{K}, \bar{K}) \cong \text{Hom}_{\bar{R}}(\bar{K}, \bar{K})$.

Because of this isomorphism we can write exact sequence (ii) in the form:

$$(iii) \quad \text{Hom}_R(K, K) \rightarrow \text{Hom}_{\bar{R}}(\bar{K}, \bar{K}) \rightarrow 0.$$

The fact that R is complete in the R -topology gives us an isomorphism $R \cong \text{Hom}_R(K, K)$, and we make this identification in (iii). Thus (iii) induces an exact sequence of \bar{R} -modules:

$$(iv) \quad \bar{R} \rightarrow \text{Hom}_{\bar{R}}(\bar{K}, \bar{K}) \rightarrow 0.$$

Since \bar{K} is a divisible \bar{R} -module, $\text{Hom}_{\bar{R}}(\bar{K}, \bar{K})$ is torsion-free as an \bar{R} -module. Therefore (iv) is really an isomorphism: $\bar{R} \cong \text{Hom}_{\bar{R}}(\bar{K}, \bar{K})$, which shows that \bar{R} is complete in the \bar{R} -topology by [8, Proposition 6.4].

Now \bar{R} has exactly two maximal ideals, and P was chosen to be maximal among the prime ideals contained in $M_1 \cap M_2$. Therefore, \bar{R} is an h -local domain. By [8, Theorem 8.5] a complete h -local domain is in fact a local ring. This contradiction proves that R must have been an h -local domain to begin with.

THEOREM 5. *Let \hat{R} be an integrally closed domain such that if S is any Prüfer ring between R and \hat{R} , then $\text{inj. dim.}_R S = 1$. Then R has property D if and only if the following conditions are satisfied:*

(1) *R is an h -local domain with at most two maximal ideals.*

(2) *R_M is a maximal valuation ring for each of the maximal ideals M of R .*

In this case R is a Prüfer ring.

Proof. If the two conditions are satisfied, then R has property D by either Theorem 2 or Theorem 3. Conversely, assume that R has property D .

Case I. Suppose the collection of valuation rings between R and Q is linearly ordered. In this case the intersection of the valuation rings between R and Q is again a valuation ring. Since R is integrally closed, R is equal to this intersection. Thus R is a valuation ring. By Theorem 2, R is a maximal valuation ring.

Case II. Suppose there exist two valuation rings W_1 and W_2 between R and Q such that $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$. By Zorn's Lemma there exists a valuation ring V_1 such that $R \subset V_1 \subset W_1$ and such that V_1 is minimal in the collection of valuation rings between R and W_1 . Similarly, there exists a valuation ring V_2 such that $R \subset V_2 \subset W_2$ and such that V_2 is minimal in the collection of valuation rings between R and W_2 .

We will prove that $R = V_1 \cap V_2$. Let V be any valuation ring between R and Q , and let $S = V \cap V_1 \cap V_2$. Suppose that there are no containment relations between any two of the valuation rings V , V_1 , V_2 . By the independence of valuations [10, Theorem 11.11], S has exactly three maximal ideals. Since S has property D by Lemma 1, S is not an h -local ring by Lemma 4. Hence there exists a nonzero prime ideal contained in the intersection of two of the maximal ideals of S . Thus at least one of $S_1 = V \cap V_1$, $S_2 = V \cap V_2$, and $S_3 = V_1 \cap V_2$ is not an h -local ring.

By [10, Theorem 11.11] each of S_1 , S_2 , and S_3 is a Prüfer ring with exactly two maximal ideals, and the localizations with respect to these maximal ideals give us back V , V_1 , and V_2 . By Theorem 2 each of these valuation rings is a maximal valuation ring. The hypothesis of the theorem guarantees that each of S_1 , S_2 , and S_3 has injective dimension one as a module over itself. Hence we can apply Lemma 6 to conclude that each of S_1 , S_2 , and S_3 is an h -local ring. This contradiction shows that there is a containment relationship between two of the rings V , V_1 , and V_2 . The minimality of V_1 and V_2 forces this relationship to be either $V_1 \subset V$, or $V_2 \subset V$, or $V_1 = V_2$.

Suppose that $V_1 = V_2$. Let $m(W_1)$ and $m(W_2)$ be the maximal ideals of W_1 and W_2 , respectively. Since $V_1 \subset W_1$ and $V_2 \subset W_2$, we have $m(W_1) \subset V_1$ and $m(W_2) \subset V_2$. The ideals of V_1 are linearly ordered, and $V_1 = V_2$. Thus either $m(W_1) \subset m(W_2)$, or $m(W_2) \subset m(W_1)$. But then either $W_2 \subset W_1$, or $W_1 \subset W_2$. This contradicts our assumption that W_1 and W_2 have no containment relationships, and so we can not have $V_1 = V_2$.

Thus we must have either $V_1 \subset V$ or $V_2 \subset V$. Since V was an arbitrary valuation ring in Q containing R , it follows that $V_1 \cap V_2$ is the intersection of all of the valuation rings containing R . Since R is integrally closed, we have $R = V_1 \cap V_2$.

As we have already seen, $R = V_1 \cap V_2$ is an h -local ring with exactly two maximal ideals, and the localizations V_1 and V_2 with respect to these maximal ideals are maximal valuation rings.

Added in proof. It has been pointed out to me by D. Lazard that the question about the existence of rings of type (1) raised in the introduction was answered by

F. K. Schmidt in *Mehrfach perfekte Körper*, Math. Ann. **108** (1933), 1–25. He proved that a field cannot be complete for two distinct discrete valuations. Mrs. Barbara Osofsky has also communicated to me a short direct proof that rings of type (1) cannot exist.

Thus Theorem 4 of this paper should read as follows:

Let R be a Noetherian integral domain. Then R has property D if and only if R is a complete local domain of Krull dimension one such that every ideal of R can be generated by two elements.

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NORTHWESTERN UNIVERSITY,
EVANSTON, ILLINOIS